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EXPLICIT SOLUTIONS FOR COLLINEAR INTERFACE CRACK PROBLEMS

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Abstract—By combining Stroh's formalism, the method of analytical continuation and the translating technique introduced in this paper, the explicit full domain solutions for the problems of collinear interface cracks between dissimilar anisotropic media have been presented. To define a proper bimaterial stress intensity factor, the full domain solution is applied to the near tip of an interface crack. The results show that the effect of material properties is totally reflected through the oscillation index ε, and the bimaterial constants D_{11}, D_{22} and D_{33} which have inverse relations with Young's moduli E_1, E_2 and shear modulus $\sqrt{G_{23}G_{31}}$, respectively. For the purpose of illustration, three typical examples are solved explicitly such as a semi-infinite interface crack, a finite interface crack and two collinear interface cracks.

INTRODUCTION

Although the problems of interface cracks between dissimilar anisotropic media have been widely studied by many researchers [for example, Gotoh (1967), Clements (1971), Willis (1971), Ting (1986), Bassani and Qu (1989), Suo (1990), Wu (1990) and Gao *et al.* (1992)], the solutions provided are usually limited to the field of the interface. Some of them [like Suo (1990)] discuss the applicability to a full domain, however no explicit solutions have been given. Very few researchers are concerned about the explicit full domain solutions primarily because their attentions are focussed upon the fracture parameters such as stress intensity factors and energy release rates which can be found by knowing the near tip solutions along the interface. However, if one is interested in applying the infinite domain solutions to the finite domain problems through some numerical approaches such as the boundary element method or finite element method, the field solutions other than those along the interface are also needed. With this intention, we derive the explicit full domain solutions by combining Stroh's formalism (Stroh, 1958), the method of analytical continuation (Muskhelishvili, 1954) and the translating technique introduced in this paper.

Among all the near tip solutions presented in the literature, important controversies seem to exist in the definition of stress intensity factors. Recently, Gao *et al.* (1992) tried to justify a proper definition from the mismatch analysis and found that only the solution proposed by Wu (1990), which conforms to the general definition suggested by Rice (1988), is consistent with the analysis of local interface mismatch near the crack tip. In this paper, from the viewpoint of the classical definition of stress intensity factors for the homogeneous media, which treats the factors as scalars measuring the intensity of singularities of the stresses near the tip, a relation between two seemingly different definitions given by Suo (1990) and Wu (1990) has been constructed. Through this relation, one can see more clearly the physical meaning of stress intensity factors and understand that the controversy between those two definitions is just a vector transformation. This also explains why the bimaterial stress intensity factors defined by Suo (1990) cannot be reduced to the classical stress intensity factors.

For the purpose of illustration, three typical examples are solved explicitly, i.e. semi-infinite interface crack, a finite interface crack and two collinear interface cracks. The results show that the stress intensity factors for orthotropic bimaterial interface cracks are strongly similar to those of isotropic bimaterials. The only difference is that the factor $\sqrt{D_{11}/D_{22}}$ may not be equal to unity, in which D_{11} and D_{22} are bimaterial constants that have inverse relations with Young's moduli E_1 and E_2, respectively.

COLLINEAR INTERFACE CRACK PROBLEMS

In my recent paper (Hwu, 1992), a general solution for the problems of thermoelastic interface cracks between dissimilar anisotropic media has been obtained by using Stroh's formalism (Stroh, 1958) and the method of analytical continuation (Muskhelishvili, 1954). Without considering the thermal effect, a general solution for the present problem (Fig. 1) can be summarized as

$$\left.\begin{aligned}\mathbf{u}_1 &= \mathbf{A}_1\mathbf{f}_1(z)+\bar{\mathbf{A}}_1\overline{\mathbf{f}_1(z)}\\ \boldsymbol{\phi}_1 &= \mathbf{B}_1\mathbf{f}_1(z)+\bar{\mathbf{B}}_1\overline{\mathbf{f}_1(z)}\end{aligned}\right\},\quad z\in S_1 \tag{1a}$$

and

$$\left.\begin{aligned}\mathbf{u}_2 &= \mathbf{A}_2\mathbf{f}_2(z)+\bar{\mathbf{A}}_2\overline{\mathbf{f}_2(z)}\\ \boldsymbol{\phi}_2 &= \mathbf{B}_2\mathbf{f}_2(z)+\bar{\mathbf{B}}_2\overline{\mathbf{f}_2(z)}\end{aligned}\right\},\quad z\in S_2, \tag{1b}$$

where the subscripts 1 and 2 are used to denote the quantities pertaining to the materials 1 and 2 which are located on $x_2 > 0$ (S_1) and $x_2 < 0$ (S_2), respectively. The overbar represents the conjugate of a complex number. $\mathbf{u}$ and $\boldsymbol{\phi}$ represent the displacement and stress function, respectively. The stresses σ_{ij} are related to the stress function $\boldsymbol{\phi}$ by

$$\sigma_{i1} = -\phi_{i,2},\quad \sigma_{i2} = \phi_{i,1}. \tag{1c}$$

$\mathbf{A}$ and $\mathbf{B}$ are 3×3 complex matrices composed of the elasticity constants. $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ are two complex function vectors which should be determined through the boundary value conditions set for the interface crack problems. The solutions have been found by Hwu (1992) as

$$\begin{aligned}\mathbf{f}_1(z) &= \mathbf{B}_1^{-1}\boldsymbol{\psi}(z),\\ \mathbf{f}_2(z) &= \mathbf{B}_2^{-1}\bar{\mathbf{M}}^{*-1}\mathbf{M}^*\boldsymbol{\psi}(z),\end{aligned} \tag{2a}$$

where

$$\boldsymbol{\psi}'(z) = \frac{1}{2\pi i}\mathbf{X}_0(z)\int_L \frac{1}{s-z}[\mathbf{X}_0^+(s)]^{-1}\hat{\mathbf{t}}(s)\,\mathrm{d}s+\mathbf{X}_0(z)\mathbf{p}_n(z). \tag{2b}$$

In the above, prime $'$ denotes differentiation with respect to its argument; the integration path L lies along the region of cracks; $\hat{\mathbf{t}}(s)$ is the self-equilibrated prescribed traction applied to the upper and lower surfaces of the crack; $\mathbf{p}_n(z)$ is an arbitrary polynomial vector of degree not higher than the number of cracks n, which may be determined by the infinity conditions and the single-value requirement of displacements. $\mathbf{M}^*$ is the bimaterial matrix defined as

$$\mathbf{M}^* = \mathbf{D}-i\mathbf{W}, \tag{3a}$$

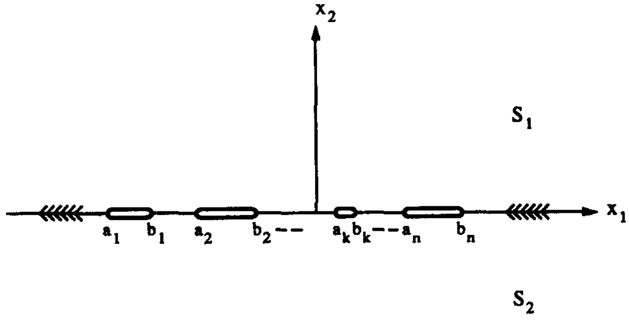

Fig. 1. Collinear interface cracks between dissimilar anisotropic media.

where

$$\mathbf{D} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1}, \quad \mathbf{W} = \mathbf{S}_1\mathbf{L}_1^{-1} - \mathbf{S}_2\mathbf{L}_2^{-1}, \tag{3b}$$

and S, L are 3×3 real matrices composed of the elasticity constants. $\mathbf{X}_0(z)$ is the basic Plemelj function matrix defined as

$$\mathbf{X}_0(z) = \mathbf{\Lambda}\mathbf{\Gamma}(z), \tag{4a}$$

where

$$\mathbf{\Lambda} = [\boldsymbol{\lambda}_1 \quad \boldsymbol{\lambda}_2 \quad \boldsymbol{\lambda}_3],$$
$$\mathbf{\Gamma}(z) = \left\langle\!\left\langle \prod_{j=1}^{n} (z-a_j)^{-(1+\delta_\alpha)}(z-b_j)^{\delta_\alpha} \right\rangle\!\right\rangle. \tag{4b}$$

The angular brackets $\langle\langle\ \rangle\rangle$ stand for the diagonal matrix, i.e.

$$\langle\langle f_\alpha \rangle\rangle = \text{diag.}\,[f_1, f_2, f_3],$$

in which each component is varied according to the Greek index α. This notation will be used throughout this paper. δ_α and $\boldsymbol{\lambda}_\alpha$, $\alpha = 1, 2, 3$, of (4b) are the eigenvalues and eigenvectors of

$$(\mathbf{M}^* + e^{2i\pi\delta}\bar{\mathbf{M}}^*)\boldsymbol{\lambda} = \mathbf{0}. \tag{5a}$$

The explicit solution for the eigenvalue δ has been given by Ting (1986) as

$$\delta_\alpha = -\tfrac{1}{2} + i\varepsilon_\alpha, \quad \alpha = 1, 2, 3, \tag{5b}$$

where

$$\varepsilon_1 = \varepsilon = \frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta}, \quad \varepsilon_2 = -\varepsilon, \quad \varepsilon_3 = 0, \quad \beta = [-\tfrac{1}{2}\mathrm{tr}\,(\mathbf{W}\mathbf{D}^{-1})^2]^{1/2}, \tag{5c}$$

and tr stands for the trace of the matrix; ε is called the oscillation index since it characterizes the oscillatory behavior of the stresses near the crack tip.

Note that the complex functions $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ given in (2) are understood to be function vectors in which each component is a function with argument z_1, z_2 and z_3, respectively, where $z_\alpha = x_1 + p_\alpha x_2$, $\alpha = 1, 2, 3$. (x_1, x_2) is a fixed rectangular coordinate system, and p_α is a material eigenvalue.

To get the explicit full domain solution, the following translating technique is introduced. If an implicit solution is written as

$$\mathbf{f}(z) = \mathbf{C} \ll g_\alpha(z) \gg \mathbf{q}, \tag{6a}$$

with the understanding that the subscript of z is dropped before the matrix product and a replacement of z_1, z_2 or z_3 should be made for each component function of $\mathbf{f}(z)$ after the multiplication of matrices, the explicit solution can be expressed as

$$\mathbf{f}(z) = \sum_{k=1}^{3} \ll g_k(z_\alpha) \gg \mathbf{C}\mathbf{I}_k\mathbf{q}, \tag{6b}$$

where

$$\mathbf{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6c}$$

According to the description of the angular bracket following (4b), $\ll g_\alpha(z) \gg$ means diag. $[g_1(z), g_2(z), g_3(z)]$ and $\ll g_k(z_\alpha) \gg$ means diag. $[g_k(z_1), g_k(z_2), g_k(z_3)]$. After finding the implicit solution from eqn (2) and the explicit solution by using the translating technique introduced in eqn (6), the explicit full field solutions for the displacements and stresses can then be obtained by the general solution shown in (1).

The application of fracture mechanics bears largely upon the stress intensity factors $\mathbf{K}$, crack opening displacements $\Delta\mathbf{u}$ and energy release rate G. To calculate these parameters, we concentrate on a crack-tip region which is small compared with the whole body but sufficiently large in terms of atomic dimensions for us to be reasonably happy with the application of linear elasticity theory. The physical problem considered is therefore a semi-infinite traction-free interface crack. By solving this problem through the use of eqns (1)–(6) and the operation of normalization, scaling, nondimensionalization and vector transformation, a comparable definition which can be reduced to the classical stress intensity factors for a crack tip in homogeneous media has been obtained as (Wu, 1990; Hwu *et al.*, 1991)

$$\mathbf{K} = \lim_{r \to 0} \sqrt{2\pi r}\, \mathbf{\Lambda} \ll (r/\ell)^{-i\varepsilon_\alpha} \gg \mathbf{\Lambda}^{-1} \boldsymbol{\phi}', \tag{7a}$$

where

$$\mathbf{K} = \begin{Bmatrix} K_{II} \\ K_I \\ K_{III} \end{Bmatrix}, \tag{7b}$$

and r is the distance from the crack tip; ℓ is a length parameter which may be chosen arbitrarily as long as it is held fixed when specimens of a given material pair are compared. Different values of ℓ will not alter the magnitude of $\mathbf{K}$ but will change its phase angle. Note that in this definition $\mathbf{\Lambda}$ has been normalized by

$$\bar{\mathbf{\Lambda}}^T \mathbf{D} \mathbf{\Lambda} = \mathbf{I}.$$

With this definition of bimaterial stress intensity factors, the near tip solution can then be expressed as (Hwu *et al.*, 1991)

$$\boldsymbol{\phi}' = \frac{1}{\sqrt{2\pi r}} \mathbf{\Lambda} \langle\langle (r/\ell)^{i\varepsilon_\alpha} \rangle\rangle \mathbf{\Lambda}^{-1} \mathbf{K},$$

$$\Delta\mathbf{u} = \sqrt{\frac{2r}{\pi}}\, \bar{\mathbf{\Lambda}}^{-T} \left\langle\left\langle \frac{(r/\ell)^{i\varepsilon_\alpha}}{(1+2i\varepsilon_\alpha)\cosh(\pi\varepsilon_\alpha)} \right\rangle\right\rangle \mathbf{\Lambda}^{-1} \mathbf{K}. \tag{8}$$

Moreover, a simple quadratic relation between the energy release rate G and stress intensity factors $\mathbf{K}$ can also be derived. That is

$$G = \tfrac{1}{4} \mathbf{K}^T \mathbf{E} \mathbf{K}, \quad \mathbf{E} = \mathbf{D} + \mathbf{W} \mathbf{D}^{-1} \mathbf{W}, \tag{9}$$

which is equivalent to the one given by Wu (1990).

It is also noted that a seemingly different definition **k** given by Suo (1990) is related to the present **K** by a vector transformation, i.e.

$$\mathbf{K} = \mathbf{\Lambda}\hat{\mathbf{k}}, \quad \hat{\mathbf{k}} = \langle\langle \ell^{i\varepsilon_\alpha} \rangle\rangle \mathbf{k}. \tag{10}$$

The near tip solutions (8) show that the influence of material properties is reflected through the eigenvalues δ_α (hence ε_α) and eigenvector matrix $\mathbf{\Lambda}$. The explicit solution for the eigenvalues δ_α has been given in (5b,c), which is related to the matrices **W** and **D**, or **S** and **L** by (3b). The explicit expressions in terms of the elasticity constants C_{ij} for **S** and **L** of orthotropic materials have been shown by Dongye and Ting (1989). Hence, it is possible to get the explicit solutions for the bimaterial matrices **W** and **D**, the eigenvalue ε and eigenvector matrix $\mathbf{\Lambda}$, and the energy release rate G in terms of the engineering constants. By direct substitution (Hwu *et al.*, 1991) it shows that for orthotropic bimaterials **D** is a diagonal matrix which is symmetric and positive definite, while **W** is antisymmetric with only one independent component W_{21} $(= -W_{12})$ which has an inverse relation with Young's modulus E_1. Furthermore, D_{11}, D_{22} and D_{33} have inverse relations with Young's moduli E_1, E_2 and shear modulus $\sqrt{G_{23}G_{31}}$ respectively. It can also be shown that the normalized $\mathbf{\Lambda}$ is independent of **W** and is only related to the bimaterial constants D_{11}, D_{22} and D_{33}.

EXAMPLES

The solution obtained in eqn (2) is valid for an arbitrary number of collinear interface cracks and an arbitrary loading condition. In the following, three important examples are presented by way of illustration. The first is a semi-infinite interface crack subjected to a point load applied to the crack surface; the second is a finite interface crack subjected to a point load or uniform load on the crack surface; the last is two collinear interface cracks under uniform loading at infinity. Following the same procedures, one may find solutions for other collinear interface crack problems.

(i) *A semi-infinite interface crack*

Let the semi-infinite planes of different materials be joined along the positive x_1-axis. A line crack is situated along the negative x_1-axis extending from $x_1 = 0$ to $x_1 = -\infty$ and is opened by a point force $\mathbf{t}_0$ at $z = -a$ on each side of the crack (Fig. 2). For this problem, the Plemelj function $\mathbf{X}_0(z)$ used is

$$\mathbf{X}_0(z) = \mathbf{\Lambda}\langle\langle z^{-1/2+i\varepsilon_\alpha} \rangle\rangle. \tag{11}$$

The point load $\hat{\mathbf{t}}(s)$ can be represented by a delta function, i.e.

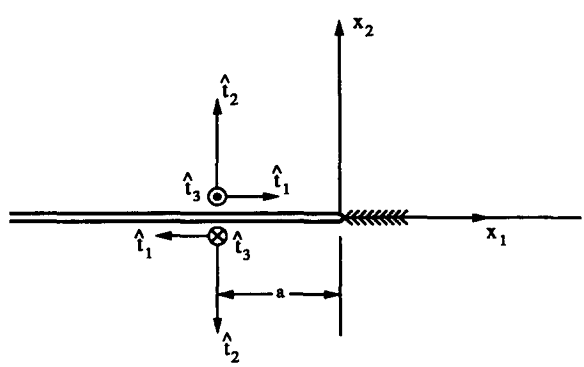

Fig. 2. A semi-infinite interface crack subjected to point loads.

$$\hat{\mathbf{t}}(s) = \delta(s+a)\mathbf{t}_0, \quad \mathbf{t}_0 = \begin{Bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \end{Bmatrix}. \tag{12}$$

Substituting (11) and (12) into (2b), we have

$$\boldsymbol{\psi}'(z) = -\frac{1}{2\pi i(z+a)}\mathbf{X}_0(z)[\mathbf{X}_0^+(-a)]^{-1}\mathbf{t}_0, \tag{13a}$$

or

$$\boldsymbol{\psi}'(z) = \frac{1}{2\pi(z+a)}\left\langle\left\langle e^{\pi\varepsilon_\alpha}\left(\frac{a}{z}\right)^{1/2-i\varepsilon_\alpha}\right\rangle\right\rangle\mathbf{\Lambda}^{-1}\mathbf{t}_0. \tag{13b}$$

With this solution, one can calculate the complex function vectors $\mathbf{f}_1(z)$ and $\mathbf{f}_2(z)$ by (2a) with the understanding described in eqn (6). The results are

$$\mathbf{f}_1(z) = \sum_{k=1}^{3}\left\langle\left\langle\frac{e^{\pi\varepsilon_k}}{2\pi}a^{1/2-i\varepsilon_k}\int\frac{z_\alpha^{-1/2+i\varepsilon_k}}{z_\alpha+a}\mathrm{d}z_\alpha\right\rangle\right\rangle\mathbf{B}_1^{-1}\mathbf{I}_k\mathbf{\Lambda}^{-1}\mathbf{t}_0,$$

$$\mathbf{f}_2(z) = \sum_{k=1}^{3}\left\langle\left\langle\frac{e^{\pi\varepsilon_k}}{2\pi}a^{1/2-i\varepsilon_k}\int\frac{z_\alpha^{*\,-1/2+i\varepsilon_k}}{z_\alpha^*+a}\mathrm{d}z_\alpha^*\right\rangle\right\rangle\mathbf{B}_2^{-1}\bar{\mathbf{M}}^{*-1}\mathbf{M}^*\mathbf{I}_k\mathbf{\Lambda}^{-1}\mathbf{t}_0. \tag{14}$$

Note that $\mathbf{f}_2(z)$ can be obtained from $\mathbf{f}_1(z)$ with z_α and $\mathbf{B}_1^{-1}$ replaced by z_α^* and $\mathbf{B}_2^{-1}\bar{\mathbf{M}}^{*-1}\mathbf{M}^*$, respectively, where $z_\alpha^* = x_1 + p_\alpha^* x_2$ and p_α^* is the eigenvalue of material 2. The bimaterial stress intensity factors defined in (7) can then be obtained by considering $z = r+a, r\to 0$ and applying the following simple relation (Hwu, 1992):

$$\boldsymbol{\phi}' = (\mathbf{I}+\bar{\mathbf{M}}^{*-1}\mathbf{M}^*)\boldsymbol{\psi}'(z). \tag{15}$$

The results are

$$\mathbf{K} = \sqrt{\frac{2}{\pi a}}\mathbf{\Lambda}\langle\langle(a/\ell)^{-i\varepsilon_\alpha}\cosh\pi\varepsilon_\alpha\rangle\rangle\mathbf{\Lambda}^{-1}\mathbf{t}_0. \tag{16}$$

For orthotropic bimaterials it can be shown that (Hwu *et al.*, 1991)

$$\mathbf{\Lambda}\langle\langle c_\alpha\rangle\rangle\mathbf{\Lambda}^{-1} = \begin{bmatrix} c_R & c_I\sqrt{D_{22}/D_{11}} & 0 \\ -c_I\sqrt{D_{11}/D_{22}} & c_R & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{17}$$

where $c_1 = c$, $c_2 = \bar{c}$, $c_3 = 1$ and c_R, c_I are real and imaginary parts of c. By using (16) and (17), the explicit solution of $\mathbf{K}$ for the orthotropic bimaterials is obtained as

$$K_\mathrm{I} = \sqrt{\frac{2}{\pi a}}\cosh\pi\varepsilon[\hat{t}_2\cos(\varepsilon\ln a/\ell)+\hat{t}_1\sqrt{D_{11}/D_{22}}\sin(\varepsilon\ln a/\ell)],$$

$$K_\mathrm{II} = \sqrt{\frac{2}{\pi a}}\cosh\pi\varepsilon[\hat{t}_1\cos(\varepsilon\ln a/\ell)-\hat{t}_2\sqrt{D_{22}/D_{11}}\sin(\varepsilon\ln a/\ell)],$$

$$K_\mathrm{III} = \sqrt{\frac{2}{\pi a}}\hat{t}_3. \tag{18}$$

This result shows that the stress intensity factors for orthotropic bimaterial interface cracks are strongly similar to those for isotropic bimaterials given by Rice and Sih (1965). The only difference is that the factor $\sqrt{D_{11}/D_{22}}$ may not be equal to unity since E_1 and E_2 are usually not the same for orthotropic materials. The results can also be reduced to the classical stress intensity factors for a crack tip in a homogeneous anisotropic medium in which $\varepsilon = 0$ by (5c) with $\mathbf{W} = \mathbf{0}$.

(ii) *A finite interface crack*

Consider an interface crack located on $a_1 = -a$, $b_1 = a$, subjected to a point force $\mathbf{t}_0$ at $z = c$ on each side of the crack [Fig. 3(a)]. For this problem, the Plemelj function $\mathbf{X}_0(z)$ is

$$\mathbf{X}_0(z) = \frac{1}{\sqrt{z^2-a^2}}\Lambda\left\langle\left\langle\left(\frac{z-a}{z+a}\right)^{i\varepsilon_\alpha}\right\rangle\right\rangle, \tag{19}$$

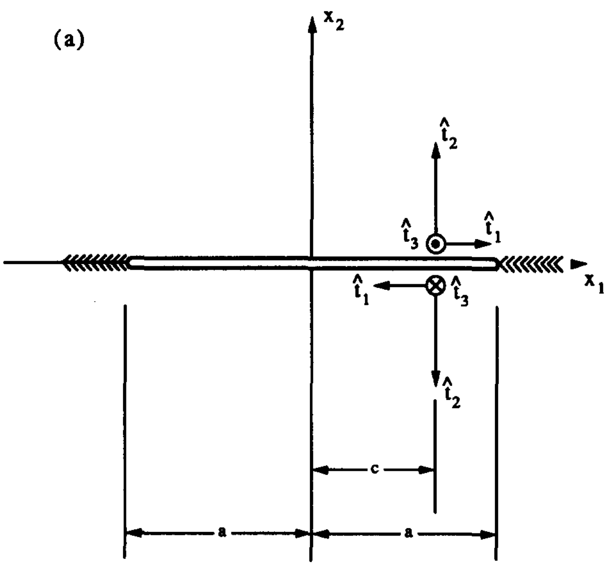

Fig. 3a. A finite interface crack subjected to point loads.

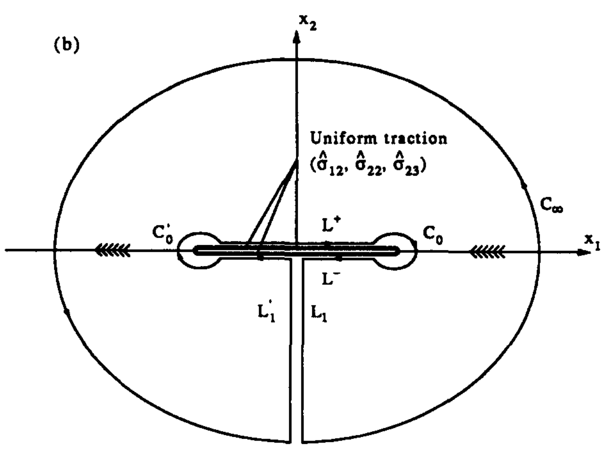

Fig. 3b. Integration contour of a finite interface crack subjected to uniform loads.

while $\hat{\mathbf{t}}(s)$ is represented by

$$\hat{\mathbf{t}}(s) = \delta(s-c)\mathbf{t}_0. \tag{20}$$

Substituting (19) and (20) into (2b), we have

$$\psi'(z) = \mathbf{A}\left\langle\left\langle \frac{e^{\pi\varepsilon_\alpha}}{2\pi}\left(\frac{a+c}{a-c}\right)^{i\varepsilon_\alpha} \frac{1}{c-z}\sqrt{\frac{a^2-c^2}{z^2-a^2}}\left(\frac{z-a}{z+a}\right)^{i\varepsilon_\alpha}\right\rangle\right\rangle \mathbf{A}^{-1}\mathbf{t}_0. \tag{21}$$

By (2a) and (6), the explicit full domain solution for $\mathbf{f}_1(z)$ is obtained as

$$\mathbf{f}_1(z) = \sum_{k=1}^{3}\left\langle\left\langle \frac{e^{\pi\varepsilon_k}}{2\pi}\left(\frac{a+c}{a-c}\right)^{i\varepsilon_k}\int \frac{1}{c-z_\alpha}\sqrt{\frac{a^2-c^2}{z_\alpha^2-a^2}}\left(\frac{z_\alpha-a}{z_\alpha+a}\right)^{i\varepsilon_k} \mathrm{d}z_\alpha\right\rangle\right\rangle \mathbf{B}_{\mathrm{I}}^{-1}\mathbf{A}\mathbf{I}_k\mathbf{A}^{-1}\mathbf{t}_0, \tag{22}$$

and $\mathbf{f}_2(z)$ can be obtained from $\mathbf{f}_1(z)$ with z_α and $\mathbf{B}_{\mathrm{I}}^{-1}$ replaced by z_α^* and $\mathbf{B}_2^{-1}\bar{\mathbf{M}}^{*-1}\mathbf{M}^*$, respectively. By a similar approach to that in (i), the bimaterial stress intensity factors of the right tip are obtained as

$$\mathbf{K} = -\frac{1}{\sqrt{\pi a}}\sqrt{\frac{a+c}{a-c}}\mathbf{A}\left\langle\left\langle\left[\frac{\ell(a+c)}{2a(a-c)}\right]^{i\varepsilon_\alpha}\cosh\pi\varepsilon_\alpha\right\rangle\right\rangle\mathbf{A}^{-1}\mathbf{t}_0. \tag{23}$$

For orthotropic bimaterials, we have

$$K_{\mathrm{I}} = -\frac{1}{\sqrt{\pi a}}\sqrt{\frac{a+c}{a-c}}\cosh\pi\varepsilon\left\{\hat{t}_2\cos\left(\varepsilon\ln\frac{2a(a-c)}{\ell(a+c)}\right)+\hat{t}_1\sqrt{D_{11}/D_{22}}\sin\left(\varepsilon\ln\frac{2a(a-c)}{\ell(a+c)}\right)\right\},$$

$$K_{\mathrm{II}} = -\frac{1}{\sqrt{\pi a}}\sqrt{\frac{a+c}{a-c}}\cosh\pi\varepsilon\left\{\hat{t}_1\sin\left(\varepsilon\ln\frac{2a(a-c)}{\ell(a+c)}\right)-\hat{t}_2\sqrt{D_{22}/D_{11}}\sin\left(\varepsilon\ln\frac{2a(a-c)}{\ell(a+c)}\right)\right\},$$

$$K_{\mathrm{III}} = -\frac{1}{\sqrt{\pi a}}\sqrt{\frac{a+c}{a-c}}\hat{t}_3. \tag{24}$$

By principle of superposition, the solution of point load problems may be used to attack all the general loading conditions with the same geometry. In the following, we will show the solutions for the case of uniform loading, i.e. $\hat{\mathbf{t}} = \{\hat{\sigma}_{12}\ \hat{\sigma}_{22}\ \hat{\sigma}_{23}\}^T$ (= constant). Instead of utilizing the results of point load problems, we evaluate the line integral (2b) directly. By residue theory, the integral around a closed contour C shown in Fig. 3(b) can be calculated as

$$\oint \frac{1}{s-z}[\mathbf{X}_0(s)]^{-1}\hat{\mathbf{t}}\,\mathrm{d}s = 2\pi i\sum_{k=1}^{n}\mathbf{r}_k, \tag{25}$$

where $\mathbf{r}_k$ are the residues of the integrand at its singular points within C. The closed contour C is the union of L^+, C_0, L^-, L_1, C_∞, L_1', C_0'. The summation of the integrals along L_1 and L_1' vanishes since they have opposite directions and the integrand across this line is continuous. The integrals around the circle C_0 and C_0' can be proved to be zero when the radii of the circles C_0 and C_0' tend to zero. By replacing the contour of C_∞ by $\mathrm{R}e^{i\psi}$ and letting $R\to\infty$, the integral around C_∞ is found to be

$$\int_{C_\infty}\frac{1}{s-z}[\mathbf{X}_0(s)]^{-1}\hat{\mathbf{t}}\,\mathrm{d}s = 2\pi i\langle\langle z+2i\varepsilon_\alpha a\rangle\rangle\mathbf{A}^{-1}\hat{\mathbf{t}}. \tag{26}$$

If we let

$$\mathbf{Y}(z) = \int_{L^+} \frac{1}{s-z} [\mathbf{X}_0^+(s)]^{-1}\,\mathrm{d}s, \tag{27a}$$

knowing that (Hwu, 1992)

$$\mathbf{X}_0^+(x_1) + \bar{\mathbf{M}}^{*-1}\mathbf{M}^*\mathbf{X}_0^-(x_1) = \mathbf{0}, \quad x_1 \in L,$$

we have

$$\int_{L^+ + L^-} \frac{1}{s-z} [\mathbf{X}_0(s)]^{-1}\hat{\mathbf{t}}\,\mathrm{d}s = \mathbf{Y}(z)[\mathbf{I} - \bar{\mathbf{M}}^{*-1}\mathbf{M}^*]\hat{\mathbf{t}}. \tag{27b}$$

The only pole which makes a contribution to the residues is at $s = z$, and the residue at that point is $[\mathbf{X}_0(z)]^{-1}\hat{\mathbf{t}}$. With the above description, we are now in a position to evaluate the line integral and the final simplified result is

$$\boldsymbol{\psi}'(z) = \mathbf{A}\left\langle\left\langle 1 - \frac{z+2i\varepsilon_\alpha a}{\sqrt{z^2-a^2}}\left(\frac{z-a}{z+a}\right)^{i\varepsilon_\alpha}\right\rangle\right\rangle \mathbf{A}^{-1}(\mathbf{I} + \bar{\mathbf{M}}^{*-1}\mathbf{M}^*)^{-1}\hat{\mathbf{t}}. \tag{28}$$

By eqns (2a) and (6), we have

$$\mathbf{f}_1(z) = \sum_{k=1}^{3}\left\langle\left\langle z_\alpha - \sqrt{z_\alpha^2-a^2}\left(\frac{z_\alpha-a}{z_\alpha+a}\right)^{i\varepsilon_k}\right\rangle\right\rangle \mathbf{B}_1^{-1}\mathbf{A}\mathbf{I}_k\mathbf{A}^{-1}(\mathbf{I} + \bar{\mathbf{M}}^{*-1}\mathbf{M}^*)^{-1}\hat{\mathbf{t}}, \tag{29}$$

and $\mathbf{f}_2(z)$ can be obtained from $\mathbf{f}_1(z)$ with z_α and $\mathbf{B}_1^{-1}$ replaced by z_α^* and $\mathbf{B}_2^{-1}\bar{\mathbf{M}}^{*-1}\mathbf{M}^*$, respectively. By a similar approach to that in (i), we have

$$\mathbf{K} = -\sqrt{\pi a}\,\mathbf{A}\langle\langle(1+2i\varepsilon_\alpha)(2a/\ell)^{-i\varepsilon_\alpha}\rangle\rangle\mathbf{A}^{-1}\hat{\mathbf{t}}. \tag{30}$$

For orthotropic bimaterials, we have

$$K_{\mathrm{I}} = -\sqrt{\pi a}\,\{\hat{\sigma}_{22}[\cos(\varepsilon\ln 2a/\ell) + 2\varepsilon\sin(\varepsilon\ln 2a/\ell)] + \hat{\sigma}_{12}\sqrt{D_{11}/D_{22}}\,[\sin(\varepsilon\ln 2a/\ell) - 2\varepsilon\cos(\varepsilon\ln 2a/\ell)]\},$$

$$K_{\mathrm{II}} = -\sqrt{\pi a}\,\{\hat{\sigma}_{12}[\cos(\varepsilon\ln 2a/\ell) + 2\varepsilon\sin(\varepsilon\ln 2a/\ell)] - \hat{\sigma}_{22}\sqrt{D_{22}/D_{11}}\,[\sin(\varepsilon\ln 2a/\ell) - 2\varepsilon\cos(\varepsilon\ln 2a/\ell)]\},$$

$$K_{\mathrm{III}} = -\sqrt{\pi a}\,\hat{\sigma}_{32}. \tag{31}$$

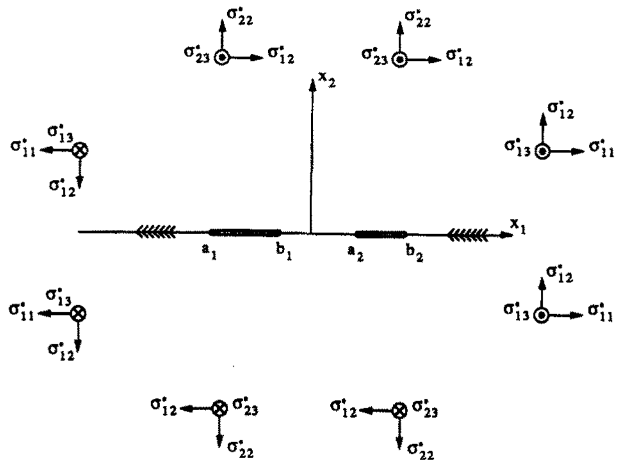

Fig. 4. Two collinear interface cracks subjected to uniform loads at infinity.

(iii) *Two collinear interface cracks*

Consider two collinear interface cracks located on (a_1, b_1) and (a_2, b_2) subjected to a uniform loading σ_{ij}^{∞} at infinity (Fig. 4). If the crack surfaces are traction free, $\hat{\mathbf{t}}(s) = \mathbf{0}$ and $\boldsymbol{\psi}'(z)$ given in (2b) can be simplified as

$$\boldsymbol{\psi}'(z) = \mathbf{X}_0(z)\mathbf{p}_2(z), \tag{32a}$$

where

$$\mathbf{X}_0(z) = \mathbf{\Lambda}\langle\langle X_\alpha(z)\rangle\rangle,$$
$$\mathbf{p}_2(z) = \mathbf{c}_2 z^2 + \mathbf{c}_1 z + \mathbf{c}_0, \tag{32b}$$

and

$$X_\alpha(z) = \frac{1}{\sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)}}\left[\frac{(z-b_1)(z-b_2)}{(z-a_1)(z-a_2)}\right]^{i\varepsilon_\alpha}. \tag{32c}$$

$\mathbf{c}_0$, $\mathbf{c}_1$ and $\mathbf{c}_2$ are coefficient vectors of the polynomial $\mathbf{p}_2(z)$, which will be determined by the infinity condition and the single-valuedness requirement. From (15) and (32), the infinity condition provides

$$\mathbf{c}_2 = \mathbf{\Lambda}^{-1}(\mathbf{I}+\bar{\mathbf{M}}^{*-1}\mathbf{M}^*)^{-1}\mathbf{t}^{\infty}, \tag{33a}$$

where

$$\mathbf{t}^{\infty} = \{\sigma_{12}^{\infty} \quad \sigma_{22}^{\infty} \quad \sigma_{32}^{\infty}\}^T. \tag{33b}$$

The requirement of the single-value condition can be expressed by

$$\int_{a_j}^{b_j} [\boldsymbol{\psi}'(x_1^+) - \boldsymbol{\psi}'(x_1^-)]\,\mathrm{d}x_1 = \mathbf{0}, \quad j = 1, 2. \tag{34}$$

To calculate $\mathbf{c}_0$ and $\mathbf{c}_1$ from the above two equations, one should evaluate $X_\alpha(z)$ along the crack surfaces which is, in a similar way to that introduced in Hwu (1991):

$$X_\alpha(z) = \pm i\,\mathrm{e}^{\mp\pi\varepsilon_\alpha}\chi_\alpha(x_1), \quad a_1 \leqslant x_1 \leqslant b_1 \quad \text{or} \quad a_2 \leqslant x_1 \leqslant b_2, \quad x_2 = 0^{\pm}, \tag{35a}$$

where

$$\chi_\alpha(x_1) = \frac{1}{\sqrt{(x_1-a_1)(b_1-x_1)(a_2-x_1)(b_2-x_1)}}\left[\frac{(b_1-x_1)(b_2-x_1)}{(x_1-a_1)(a_2-x_1)}\right]^{i\varepsilon_\alpha}, \quad a_1 \leqslant x_1 \leqslant b_1,$$

$$\chi_\alpha(x_1) = \frac{-1}{\sqrt{(x_1-a_1)(x_1-b_1)(x_1-a_2)(b_2-x_1)}}\left[\frac{(x_1-b_1)(b_2-x_1)}{(x_1-a_1)(x_1-a_2)}\right]^{i\varepsilon_\alpha}, \quad a_2 \leqslant x_1 \leqslant b_2. \tag{35b}$$

Substituting (35) into (34) and solving a system of linear algebraic equations, one may obtain

$$\mathbf{c}_0 = \langle\langle \lambda_{3\alpha}/\lambda_{1\alpha}\rangle\rangle \mathbf{c}_2, \quad \mathbf{c}_1 = \langle\langle \lambda_{2\alpha}/\lambda_{1\alpha}\rangle\rangle \mathbf{c}_2, \tag{36a}$$

where

$$\lambda_{1\alpha} = -\int_{a_1}^{b_1} \chi_\alpha(x_1)\,dx_1 \int_{a_2}^{b_2} x_1\chi_\alpha(x_1)\,dx_1 + \int_{a_1}^{b_1} x_1\chi_\alpha(x_1)\,dx_1 \int_{a_2}^{b_2} \chi_\alpha(x_1)\,dx_1,$$

$$\lambda_{2\alpha} = \int_{a_1}^{b_1} \chi_\alpha(x_1)\,dx_1 \int_{a_2}^{b_2} x_1^2\chi_\alpha(x_1)\,dx_1 - \int_{a_1}^{b_1} x_1^2\chi_\alpha(x_1)\,dx_1 \int_{a_2}^{b_2} \chi_\alpha(x_1)\,dx_1,$$

$$\lambda_{3\alpha} = \int_{a_1}^{b_1} x_1^2\chi_\alpha(x_1)\,dx_1 \int_{a_2}^{b_2} x_1\chi_\alpha(x_1)\,dx_1 - \int_{a_1}^{b_1} x_1\chi_\alpha(x_1)\,dx_1 \int_{a_2}^{b_2} x_1^2\chi_\alpha(x_1)\,dx_1. \tag{36b}$$

Combining (32), (33) and (36), the final simplified result for $\psi'(z)$ is

$$\psi'(z) = \mathbf{A}\langle\langle \frac{1}{\lambda_{1\alpha}}(\lambda_{1\alpha}z^2+\lambda_{2\alpha}z+\lambda_{3\alpha})X_\alpha(z)\rangle\rangle \mathbf{A}^{-1}(\mathbf{I}+\bar{\mathbf{M}}^{*-1}\mathbf{M}^*)^{-1}\mathbf{t}^\infty. \tag{37}$$

By (2a) and (6), the explicit full domain solution for $\mathbf{f}_1(z)$ is obtained as

$$\mathbf{f}_1(z) = \sum_{k=1}^{3}\left\langle\left\langle \int \frac{1}{\lambda_{1k}}(\lambda_{1k}z_\alpha^2+\lambda_{2k}z_\alpha+\lambda_{3k})X_k(z_\alpha)\right\rangle\right\rangle \mathbf{B}_1^{-1}\mathbf{A}\mathbf{I}_k\mathbf{A}^{-1}(\mathbf{I}+\bar{\mathbf{M}}^{*-1}\mathbf{M}^*)^{-1}\mathbf{t}^\infty, \tag{38}$$

and $\mathbf{f}_2(z)$ can be obtained from $\mathbf{f}_1(z)$ with z_α and $\mathbf{B}_1^{-1}$ replaced by z_α^* and $\mathbf{B}_2^{-1}\bar{\mathbf{M}}^{*-1}\mathbf{M}^*$, respectively. By a similar approach to that in (i), the bimaterial stress intensity factors are obtained as

$$\mathbf{K} = \sqrt{2\pi}\,\mathbf{A}\,\langle\langle k_\alpha\rangle\rangle \mathbf{A}^{-1}\mathbf{t}^\infty, \tag{39a}$$

where

$$k_\alpha = \frac{\lambda_{1\alpha}a_1^2+\lambda_{2\alpha}a_1+\lambda_{3\alpha}}{\lambda_{1\alpha}\sqrt{(b_1-a_1)(a_2-a_1)(b_2-a_1)}}\left[\frac{(b_1-a_1)(b_2-a_1)}{\ell(a_2-a_1)}\right]^{i\varepsilon_\alpha}, \quad x_1 = a_1,$$

$$= \frac{\lambda_{1\alpha}b_1^2+\lambda_{2\alpha}b_1+\lambda_{3\alpha}}{\lambda_{1\alpha}\sqrt{(b_1-a_1)(a_2-b_1)(b_2-b_1)}}\left[\frac{\ell(b_2-b_1)}{(b_1-a_1)(a_2-b_1)}\right]^{i\varepsilon_\alpha}, \quad x_1 = b_1,$$

$$= \frac{\lambda_{1\alpha}a_2^2+\lambda_{2\alpha}a_2+\lambda_{3\alpha}}{\lambda_{1\alpha}\sqrt{(a_2-a_1)(a_2-b_1)(b_2-a_2)}}\left[\frac{(a_2-b_1)(b_2-a_2)}{\ell(a_2-a_1)}\right]^{i\varepsilon_\alpha}, \quad x_1 = a_2,$$

$$= \frac{\lambda_{1\alpha}b_2^2+\lambda_{2\alpha}b_2+\lambda_{3\alpha}}{\lambda_{1\alpha}\sqrt{(b_2-a_1)(b_2-b_1)(b_2-a_2)}}\left[\frac{\ell(b_2-b_1)}{(b_2-a_1)(b_2-a_2)}\right]^{i\varepsilon_\alpha}, \quad x_1 = b_2. \tag{39b}$$

It should be noted that when calculating the stress intensity factors of the left tip, i.e. $x_1 = a_1$ or a_2, the definition given in (7), which is defined based upon the horizontal right tip condition, should be modified as

$$\mathbf{K} = \lim_{r \to 0} \sqrt{2\pi r}\, \mathbf{\Lambda} \langle\langle (r/\ell)^{i\varepsilon_\alpha} \rangle\rangle \mathbf{\Lambda}^{-1} \boldsymbol{\phi}'.$$

Otherwise, a coordinate transformation should be employed in order to be consistent with the condition of the original definition.

By use of (17), the explicit solution of **K** for orthotropic bimaterials can be obtained, which is similar to those of cases (i) and (ii), and will not be presented here. The results shown in (39) with $\varepsilon = 0$ are identical to those presented in Hwu (1991) for two collinear cracks in homogeneous anisotropic media. The elastic interaction between these two interface cracks can then be studied based upon this result.

CONCLUSIONS

Based upon Stroh's formalism and the method of analytical continuation, a general formula for the collinear interface crack problems has been given in this paper. A technique which can translate the implicit solution into the explicit full domain solution has also been provided. Through this general formula and translating technique, the explicit full field solutions for the stresses and displacements can be obtained. By considering the near tip solution of semi-infinite traction-free interface crack problems, a proper definition of the bimaterial stress intensity factors is provided in this paper. The results show that the effect of material properties is totally reflected through the oscillation index ε, and the bimaterial constants D_{11}, D_{22} and D_{33} which have inverse relations with E_1, E_2 and $\sqrt{G_{23}G_{31}}$, respectively. Moreover, the stress intensity factors for the orthortopic bimaterials are strongly similar to those of isotropic bimaterials. The only difference is that the factor $\sqrt{D_{11}D_{22}}$ may not be equal to unity since Young's moduli E_1 and E_2 are not usually the same for orthotropic materials.

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